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On Probability Domains

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Abstract Motivated by *IF*-probability theory (intuitionistic fuzzy), we study n-component probability domains in which each event represents a body of competing components and the range of a state represents a simplex S_n of n-tuples of possible rewards–the sum of the rewards is a number from $[0, 1]$. For $n = 1$ we get fuzzy events, for example a bold algebra, and the corresponding fuzzy probability theory can be developed within the category *ID* of D-posets (equivalently effect algebras) of fuzzy sets and sequentially continuous D-homomorphisms. For $n = 2$ we get *IF*-events, i.e., pairs (μ, ν) of fuzzy sets $\mu, \nu \in [0, 1]^X$ such that $\mu(x) + \nu(x) \leq 1$ for all $x \in X$, but we order our pairs (events) coordinatewise. Hence the structure of *IF*-events (where $(\mu_1, \nu_1) \leq (\mu_2, \nu_2)$ whenever $\mu_1 \leq \mu_2$ and $\nu_2 \leq \nu_1$) is different and, consequently, the resulting *IF*-probability theory models a different principle. The category *ID* is cogenerated by $I = [0, 1]$ (objects of *ID* are subobjects of powers I^X), has nice properties and basic probabilistic notions and constructions are categorical. For example, states are morphisms. We introduce the category S_nD cogenerated by $S_n = \{(x_1, x_2, \dots, x_n) \in I^n;\ \sum_{i=1}^n x_i \leq 1\}$ carrying the coordinatewise partial order, difference, and sequential convergence and we show how basic probability notions can be defined within S_nD.

Keywords Probability domain · Generalized probability measure · D-poset of fuzzy sets · Bold algebra · Cogenerator · S_nD-domain · S_nD-homomorphism · S_nD-measurable map · Duality · S_nD-probability

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Introduction

In this section we provide some background information and recall some notions used in the sequel.

D-posets have been introduced in [14] in order to model events in quantum probability. They generalize *MV*-algebras and other probability domains and provide a category in which observables and states become morphisms. The structure of an *MV*-algebra is well understood and there is an extensive literature from which an interested reader can get information about applications (cf. [6]). In our paper we deal with bold algebras, i.e. *MV*-algebras of functions ranging in [0, 1] (hence special systems of fuzzy sets). Detailed definition and relevant information can be found in [8]. Recall that a *D*-poset is a partially ordered set with the greatest element 1, the least element 0, and a partial binary operation called *difference*, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for each $a \in X$;
(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

Fundamental to applications [10] are *D*-posets of fuzzy sets, i.e. systems $\mathcal{X} \subseteq I^X$ carrying the coordinatewise partial order, coordinatewise convergence of sequences, containing the top and bottom elements of I^X, and closed with respect to the partial operation difference defined coordinatewise. Denote *ID* the category having *D*-posets of fuzzy sets as objects and having sequentially continuous *D*-homomorphisms as morphisms. Objects of *ID* are subobjects of the powers I^X.

An interested reader can find a detailed information about the *IF*-probability and its applications in [2, 19]. Here we only point out some of the basic features, mainly to stress the differences between the *IF*-probability and the new concept of S_nD-probability.

Each event A in the *IF*-probability is a pair (μ_A, ν_A) of fuzzy subsets $\mu_A, \nu_A \in [0,1]^X$ such that $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$, where μ_A is called the membership and ν_A the nonmembership function. In a sense, $\mu_A(x) + \nu_A(x) \leq 1$ is related to the intuitionistic fuzzy logic, where the Law of Excluded Middle does not hold. The *IF*-events are partially ordered: $(\mu_B, \nu_B) \leq (\mu_A, \nu_A)$ whenever $\mu_B \leq \mu_A$ and $\nu_A \leq \nu_B$. The *IF*-probability measure sends *IF*-events to the closed subintervals of $I = [0, 1]$.

Consider $S_2 = \{x = (x_1, x_2) \in I^2;\ x_1 + x_2 \leq 1\}$ carrying the coordinatewise partial order ($y \leq x$ iff $y_1 \leq x_1$ and $y_2 \leq x_2$) and a partial operation "difference" defined as follows: $x \ominus y = (x_1 - y_1, x_2 - y_2)$ iff $y \leq x$. For a nonempty set X, let S_2^X be the set of all maps of X into S_2 carrying the coordinatewise partial order and the coordinatewise "difference": $(u \ominus v)(x) = (u_1(x) \ominus v_1(x), u_2(x) \ominus v_2(x))$, $x \in X$, whenever $u = (u_1, u_2)$, $v = (v_1, v_2) \in S_2^X$ and $v(x) \leq u(x)$ for all $x \in X$. If X is a singleton $\{a\}$, then we identify S_2 and $S_2^{\{a\}}$.

Even though *IF*-events and the elements of S_2^X are pairs of fuzzy sets (the sum of the corresponding membership functions is bounded by 1), the resulting concepts are different. Also, as shown in [18, 19], the *IF*-probability can utilize *MV*-algebra technique, while the S_2D-probability is a special case of a theory based on an n-dimensional cogenerator S_n and utilizes a generalization of the partial difference operation developed by F. Kôpka and F. Chovanec (cf. [5, 10, 14–17]).

It is natural to start with a distinguished system $\mathcal{A} \subseteq I^X$ and consider systems $\mathcal{X} \subseteq S_2^X$ such that $u = (u_1, u_2) \in \mathcal{X}$ iff $u_1, u_2 \in \mathcal{A}$ and $u_1(x) + u_2(x) \leq 1$ for all $x \in X$. For example, $\mathcal{A}$ can be a bold algebra of suitable measurable functions or an *ID*-poset. Then it is possible to develop a duality theory (also for S_n) between "structure preserving maps" of $\mathcal{Y} \subseteq S_2^Y$ into $\mathcal{X} \subseteq S_2^X$ (leading to S_2D-observables) and "measurable maps" of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$ (leading to S_2D-random variables) so that it generalizes the duality theory for ID as developed in [15] and [9] (leading to the duality between observables and fuzzy random variables).

In the present paper we introduce the category S_nD cogenerated by $S_n = \{(x_1, x_2, \dots, x_n) \in I^n;\ \sum_{i=1}^n x_i \leq 1\}$ carrying the coordinatewise partial order, difference, and sequential convergence (essentially, the objects of S_nD are subobjects of the powers S_n^X) and we show how basic probability notions can be defined within S_nD. In the resulting S_nD-probability we have n-component probability domains in which each event represents a body of competing components and the range of a state represents a simplex S_n of n-tuples of possible "rewards"—the sum of the rewards is a number from [0, 1]. For $n = 1$ we get fuzzy events and the corresponding fuzzy probability theory (cf. [10, 12, 16]).

Concerning the undefined notions, the reader is referred to [6] and [1].

1 On Probability Domains

In this section we discuss some general properties of probability domains. In fact, we have rather in mind mathematical properties and we avoid the other side of the topic, that is experiments, measurements, philosophy, etc.

Question 1.1 What are probability events?

Answer (one of the possible)

- Start with a "system $\mathcal{A}$ of events";
- Choose a "cogenerator C"—usually a structured set suitable for "measuring" (e.g., the two-element Boolean algebra {0,1}, the interval $I = [0, 1]$ carrying the Łukasiewicz MV-structure, D-poset structure, ...);
- Choose a set X of "properties" measured via C so that X separates $\mathcal{A}$;
- Represent each event $a \in \mathcal{A}$ via the "evaluation" of $\mathcal{A}$ into C^X sending $a \in \mathcal{A}$ to $a_X \in C^X$, $a_X \equiv \{x(a);\ x \in X\}$ (see Fig. 1);
- Form the minimal "subalgebra" D of C^X containing $\{a_X; a \in \mathcal{A}\}$ (see Fig. 2);

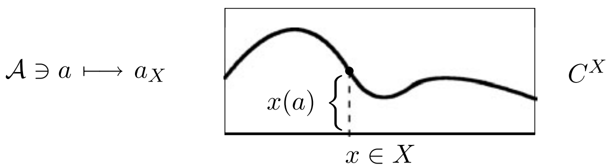

Fig. 1 Representation of an event via evaluation

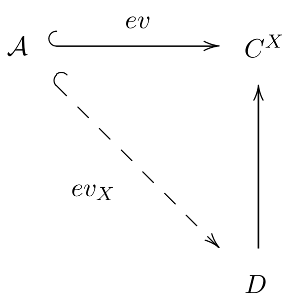

Fig. 2 From events to a domain

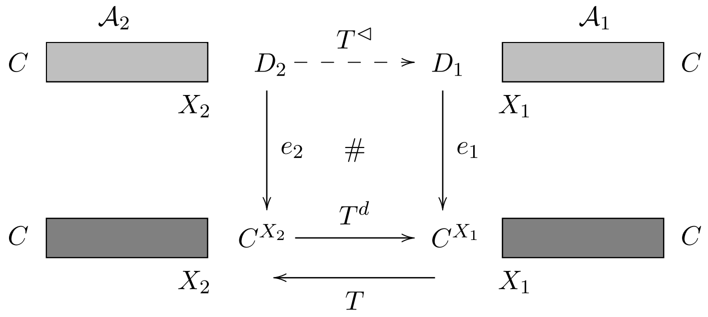

Fig. 3 Domains as objects

- The subalgebra forms a *probability domain* $D \subseteq C^X$ which has nice categorical properties.

Fields of sets, bold algebras and *ID*-posets can serve as typical examples of probability domains described above.

Question 1.2 What are elementary events?

Answer (one of the possible) The set X of characteristic properties of events.

Question 1.3 What is an observable?

Answer A suitable "structure preserving map" of one probability domain into another one. The image of the former a subdomain of the latter.

Question 1.4 How are observables related to the characteristic properties?

Answer If $D_i \subseteq C^{X_i}$, $i = 1, 2$, then each map $T : X_1 \to X_2$ induces the dual map T^d of C^{X_2} into C^{X_1} defined for $a \in C^{X_2}$ by $(T^d(a))(x) = a(T(x))$, $x \in X_1$. If T^d restricted to $D_2 \subseteq C^{X_2}$ yields a morphism $T^{\triangleleft}$ into D_1, then T is a "meaningful" transformation of X_1 into X_2 (see Fig. 3); it will be called *measurable*.

Question 1.5 What is a generalized probability measure (state)?

Answer A suitable "structure preserving map" of the probability domain D into C.

Our approach to probability domains can be summarized as follows;

Observation 1.6 The cogenerator C is the range of states and probability domains carry the initial structure with respect to states. Basic constructions are categorical.

Observation 1.7 For $C = [0, 1]$ considered as a D-poset, the classical probability domains (fields of sets) and fuzzy probability domains (bold algebras) become special cases.

Observation 1.8 Nontraditional cogenerators provide nontraditional models of probability theory.

2 S_nD-Posets

In this section we introduce the category S_nD and describe its basic properties. For $n \in \{1, 2, \dots\}$ denote $S_n = \{(x_1, x_2, \dots, x_n) \in I^n; \sum_{i=1}^n x_i \le 1\}$ carrying the coordinatewise partial order, difference and sequential convergence. Let X be a nonempty set and let ${S_n}^X$ be the set of all maps of X into S_n; if X is a singleton $\{a\}$, then $S_n^{\{a\}}$ will be condensed to S_n. Let $\mathbf{f} \in {S_n}^X$. Then there are n maps $f_1, f_2, \dots, f_n$ of X into I such that for each $x \in X$ we have $\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_n(x))$; we shall write $\mathbf{f} = (f_1, f_2, \dots, f_n)$. In what follows, ${S_n}^X$ carries the coordinatewise partial order, the coordinatewise partial difference, and the coordinatewise sequential convergence inherited from S_n. Elements $(f_1, f_2, \dots, f_n) \in {S_n}^X$ such that $\sum_{i=1}^n f_i(x) = 1$, $x \in X$, are maximal. If for some index i, $1 \le i \le n$, we have $f_j(x) = 0$ for all $j \ne i$ and all $x \in X$, then $(f_1, f_2, \dots, f_n)$ is said to be *pure*; denote $\mathbf{p_i}$ the corresponding maximal pure element of ${S_n}^X$. Clearly, if for all i, $1 \le i \le n$, the functions f_i are constant zero functions, then $(f_1(x), f_2(x), \dots, f_n(x))$ is the least element of ${S_n}^X$; it is called the *bottom* element and denoted by $\mathbf{b}$. To avoid complicated notation, if no confusion can arise, then the bottom elements, resp. the ith maximal pure elements, will be denoted by the same symbol $\mathbf{b}$, resp. $\mathbf{p_i}$, $1 \le i \le n$, independently of the ground set X.

Let X be a nonempty set. We are interested in subsets $\mathcal{X} \subseteq {S_n}^X$ closed with respect to the difference, containing the bottom element and all maximal pure elements of ${S_n}^X$. For $n = 1$ we get D-posets of fuzzy sets and for $n > 1$ we get a structure which generalizes fuzzy events to higher dimensions.

Let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n \subseteq I^X$ be reduced *ID*-posets. Define $S(\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n)$ to be the set of all $(f_1, f_2, \dots, f_n) \in {S_n}^X$ such that $f_i \in \mathcal{B}_i$, $1 \le i \le n$. If there exists an *ID*-poset $\mathcal{B} \subseteq I^X$ such that $\mathcal{B} = \mathcal{B}_i$, $1 \le i \le n$, then $S(\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n)$ is condensed to $S_n(\mathcal{B})$. In applications we consider only the later case.

Definition 2.1 Let X be a nonempty set. Let $\mathcal{X}$ be a subset of ${S_n}^X$, carrying the coordinatewise order, the coordinatewise convergence and closed with respect to the inherited difference. Assume that $\mathcal{X}$ contains the bottom element and all maximal pure elements. Then $(\mathcal{X}, \le, \ominus, \mathbf{b}, \mathbf{p_1}, \dots, \mathbf{p_n})$ is said to be an *S_nD-domain*. If there is a reduced *ID*-poset $\mathcal{B} \subseteq I^X$ such that $\mathcal{X} = S_n(\mathcal{B})$, then $(\mathcal{X}, \le, \ominus, \mathbf{b}, \mathbf{p_1}, \dots, \mathbf{p_n})$ is said to be a *simple S_nD-domain* and $\mathcal{B}$ is said to be the *base* of $\mathcal{X}$.

If no confusion can arise, then $(\mathcal{X}, \le, \ominus, \mathbf{b}, \mathbf{p_1}, \dots, \mathbf{p_n})$ will be reduced to $\mathcal{X}$. *In what follows, all S_nD-domains are assumed to be simple.* Clearly, each S_n^X is a simple S_nD-domain.

Definition 2.2 Let h be a map of an S_nD-domain $\mathcal{Y}$ into an S_nD-domain $\mathcal{X}$ such that

(i) $h(\mathbf{v}) \le h(\mathbf{u})$ whenever $\mathbf{u}, \mathbf{v} \in \mathcal{Y}$ and $\mathbf{v} \le \mathbf{u}$, and then $h(\mathbf{u} \ominus \mathbf{v}) = h(\mathbf{u}) \ominus h(\mathbf{v})$;
(ii) h maps the bottom element of $\mathcal{Y}$ to the bottom element of $\mathcal{X}$ and the ith maximal pure element of $\mathcal{Y}$ to the ith maximal pure element of $\mathcal{X}$, for all i, $1 \le i \le n$.

Then h is said to be an *S_nD-homomorphism.*

Example 2.3 Let $\mathcal{B} \subseteq I^X$ be a bold algebra and let $S_n(\mathcal{B}) \subseteq S_n^X$ be the corresponding simple S_nD-domain. Let s be a state on $\mathcal{B}$. For $\mathbf{f} = (f_1, f_2, \dots, f_n) \in S_n(\mathcal{B})$ define $s_n(\mathbf{f}) = (s(f_1), s(f_2), \dots, s(f_n))$. It is easy to see that s_n is a sequentially continuous S_nD-homomorphism of $S_n(\mathbf{B})$ into S_n; we shall say that s_n is based on s.

Definition 2.4 Let $\mathcal{X} \subseteq S_n^X$ be a (simple) S_nD-domain. Then $(X, \mathcal{X})$ is said to be an *S_nD-measurable space*. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be S_nD-measurable spaces and let $T: X \to Y$ be a map such that $\mathbf{v} \circ T \in \mathcal{X}$ for each $\mathbf{v} \in \mathcal{Y}$. Then T is said to be a *S_nD-measurable map* of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$.

Denote MS_nD the category of S_nD-measurable spaces and measurable maps.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be S_nD-measurable spaces, let $T : X \to Y$ be a measurable map of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$, and let T^d be the dual map of S_n^Y into S_n^X. Denote $T^\triangleleft$ the restriction of T^d to $\mathcal{Y}$ defined by $T^\triangleleft(\mathbf{v}) = \mathbf{v} \circ T$, $\mathbf{v} \in \mathcal{Y}$. Then $T^\triangleleft$ maps $\mathcal{Y}$ into $\mathcal{X}$.

Lemma 2.5 *$T^\triangleleft$ is a sequentially continuous S_nD-homomorphism.*

Proof 1. To prove that $T^\triangleleft$ is a sequentially continuous S_nD-homomorphism, assume that a sequence $\{\mathbf{v}_k\}_{k=1}^\infty$ converges to $\mathbf{v}$ in $\mathcal{Y}$, i.e., for each $y \in Y$ we have $\mathbf{v}(y) = \lim_{k\to\infty} \mathbf{v}_k(y)$. Then, for each $x \in X$, we have $(T^\triangleleft(\mathbf{v}))(x) = \mathbf{v}(T(x)) = \lim_{k\to\infty} \mathbf{v}_k(T(x))$ $= \lim_{k\to\infty}(T^\triangleleft(\mathbf{v}_k))((x))$, i.e., the sequence $\{T^\triangleleft(\mathbf{v}_k)\}_{k=1}^\infty$ converges in $\mathcal{X}$ to $T^\triangleleft(\mathbf{v})$. Hence $T^\triangleleft$ is sequentially continuous.

2. To prove that $T^\triangleleft$ is an S_nD-homomorphism we have to prove that $T^\triangleleft$ satisfies conditions (i) and (ii) of Definition 2.2.

(i) Let $\mathbf{u}, \mathbf{v} \in \mathcal{Y}, \mathbf{v} \leq \mathbf{u}$, i.e., for each $y \in Y$ we have $\mathbf{v}(y) \leq \mathbf{u}(y)$. Then for each $x \in X$ we have $(T^\triangleleft(\mathbf{v}))(x) = \mathbf{v}(T(x)) \leq \mathbf{u}(T(x)) = (T^\triangleleft(\mathbf{u}))(x)$ and $(T^\triangleleft(\mathbf{u} \ominus \mathbf{v}))(x) = (\mathbf{u} \ominus \mathbf{v})(T(x)) = (\mathbf{u}(T(x)) \ominus (\mathbf{v}(T(x))) = (T^\triangleleft(\mathbf{u}))(x) \ominus (T^\triangleleft(\mathbf{v}))(x)$.

(ii) follows directly from $T^\triangleleft(\mathbf{v}) = \mathbf{v} \circ T, \mathbf{v} \in \mathcal{Y}$. □

Question 2.6 Has each sequentially continuous S_nD-homomorphism the form of $T^\triangleleft$ for some measurable map T?

Let $\mathcal{X} \subseteq S_n^X$ be an S_nD-domain. Then each $x \in X$ can be considered as a sequentially continuous S_nD-homomorphism ev_x of $\mathcal{X}$ into S_n defined by $ev_x(\mathbf{u}) = \mathbf{u}(x), \mathbf{u} \in \mathcal{X}$. Denote X^* the set of all sequentially continuous S_nD-homomorphisms of $\mathcal{X}$ into S_n. For $\mathbf{u} \in \mathcal{X}$ put $h(\mathbf{u}) = \{ev_x(\mathbf{u}); x \in X^*\} \in I^{X^*}$ and $\mathcal{X}^* = \{h(\mathbf{u}); \mathbf{u} \in \mathcal{X}\} \subseteq I^{X^*}$. Denote h the corresponding map of $\mathcal{X}$ into $\mathcal{X}^*$. The proof of the next lemma follows directly from the construction of $\mathcal{X}^*$ and it is omitted.

Lemma 2.7

(i) *$\mathcal{X}^*$ is an S_nD-domain*;
(ii) *h is an S_nD-isomorphism of $\mathcal{X}$ onto $\mathcal{X}^*$*;
(iii) *X^* is the set of all S_nD-homomorphisms of $\mathcal{X}^*$ into S_n*;
(iv) *$(X^*)^* = X^*$ and $(\mathcal{X}^*)^* = \mathcal{X}^*$.*

Definition 2.8 Let $\mathcal{X} \subseteq {S_n}^X$ be an S_nD-domain. Then $\mathcal{X}^*$ is said to be the *sobrification* of $\mathcal{X}$. If $X = X^*$, then $\mathcal{X}$ and $(X, \mathcal{X})$ are said to be *sober*.

Lemma 2.9 *Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be sober S_nD-measurable spaces and let h be a sequentially continuous S_nD-homomorphism of $\mathcal{Y}$ into $\mathcal{X}$. Then there is a unique S_nD-measurable map T of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$ such that $T^\triangleleft = h$.*

Proof Let $x \in X$ and let ev_x be the corresponding evaluation S_nD-homomorphism of $\mathcal{X}$ into S_n defined by $ev_x(\mathbf{u}) = \mathbf{u}(x), \mathbf{u} \in \mathcal{X}$. Then the composition $ev_x \circ h$ is a sequentially continuous S_nD-homomorphism of $\mathcal{Y}$ into S_n and hence there exists a unique $y \in Y = Y^*$ such that $ev_y = ev_x \circ h$. Denote $y = T(x)$ and let T be the corresponding map of $X = X^*$ into $Y = Y^*$. Let T^d be the dual map of S_n^Y into S_n^X. Let $\mathbf{v} \in \mathcal{Y}$. Then $(h(\mathbf{v}))(x) = (ev_x \circ h)(\mathbf{v}) = (T(x))(\mathbf{v}) = \mathbf{v}(T(x)) = (T^d(\mathbf{v}))(x)$ for all $x \in V$. Thus $T^d(\mathbf{v}) = h(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{Y}$. Let U be a measurable map of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$ such that $U(x) \neq T(x)$ for some $x \in \mathcal{X}$. Let $\mathcal{B} \subseteq I^Y$ be the base of $\mathcal{Y}$. Since we assume that $\mathcal{B}$ is a reduced *ID*-poset, there exists $v \in \mathcal{B}$ such that $v(T(x)) \neq v(T(x))$. Then for $\mathbf{f} = (v, 0, \dots, 0) \in \mathcal{Y}$ we have $\mathbf{f}(U(x)) = (U^{\triangleleft}(\mathbf{f}))(x) \neq (h(\mathbf{f}))(x) = (T^{\triangleleft}(\mathbf{f}))(x) = \mathbf{f}(T(x))$. Hence $U^{\triangleleft} \neq h$ and T is uniquely determined. □

Denote S_nD the category of S_nD-domains and sequentially continuous S_nD-homomorphisms and denote SS_nD its full subcategory consisting of sober objects. Denote SMS_nD the full subcategory of MS_nD consisting of sober S_nD-measurable spaces.

Theorem 2.10 *Categories SS_nD and SMS_nD are dually isomorphic.*

Proof Clearly, Lemmas 2.5 and 2.9 yield a pair of contravariant functors: F of SMS_nD onto SS_nD sending each sober measurable space $(X, \mathcal{X})$ to the sober S_nD-domain $\mathcal{X}$ and each measurable map T between sober spaces to the sequentially continuous S_nD-homomorphism $T^{\triangleleft}$, and G of SS_nD onto SMS_nD sending a sober S_nD-domain $\mathcal{X} \subseteq S_n^X$ to the sober measurable space $(X, \mathcal{X})$ and each sequentially continuous S_nD-homomorphism h between sober S_nD-domains to the uniquely determined measurable space T such that $T^{\triangleleft} = h$. The pair yields the desired dual isomorphism. □

The next result formalizes (cf. [7]) the relationship between sequentially continuous S_nD-homomorphisms and S_nD-measurable maps and hence the relationship between generalized observables and generalized random variables in S_nD-probability theory.

Corollary 2.11 *Categories S_nD and SMS_nD are dually naturally equivalent.*

Remark 2.12 In literature, probability measures, states, and observables are usually defined as maps sequentially continuous with respect to the monotone convergence. We believe that the monotone convergence should be replaced by the coordinatewise convergence which, as the initial convergence in the cogenerated categories, is more natural.

3 Applications

In this section we show how it is possible to generalize some notions and constructions of fuzzy probability theory (cf. [3, 4, 10, 12]) in the realm of S_nD.

Let $(\Omega, \mathbf{A}, p)$ be a probability space in the classical Kolmogorov sense (i.e. Ω is a set, $\mathbf{A}$ is a σ-field of subsets of Ω, and p is a probability measure on $\mathbf{A}$). A measurable map f of Ω into the real line R, called *random variable*, sends p into a probability measure p_f, called the *distribution* of f, defined on the real Borel sets $\mathbf{B}_R$ via $p_f(B) = p(f^{\leftarrow}(B))$, $B \in \mathbf{B}_R$. In fact, f induces a map sending probability measures $\mathcal{P}(\mathbf{A})$ on $\mathbf{A}$ into probability measures $\mathcal{P}(\mathbf{B}_R)$ on $\mathbf{B}_R$ (each point $\omega \in \Omega$, or $r \in R$ is considered as a degenerated point probability measure). The preimage map $f^{\leftarrow}$, called *observable*, maps $\mathbf{B}_R$ into $\mathbf{A}$. Points of Ω are called *elementary events*, sets in $\mathbf{A}$ are called *sample random events* and sets in $\mathbf{B}_R$ are called *real* *random events*. Each random variable f can be viewed as a channel through which the probability p of the original probability space is transported to the distribution p_f, a probability measure on the real Borel sets and hence, in fact, a channel through which the probability measures on the sample random events are transported to the probability measures on the real random events; observe that each degenerated point probability measure is transported to a degenerated point probability measure.

In the fuzzy probability theory we consider a more general situation. Let $(X, \mathbf{A})$, $(Y, \mathbf{B})$ be classical measurable spaces, let $\mathcal{P}(\mathbf{A})$ be probability measures on $\mathbf{A}$, let $\mathcal{P}(\mathbf{B})$ be probability measures on $\mathbf{B}$, and let T be a map of $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ satisfying a natural measurability condition which guarantees the existence of a dual map $T^{\triangleleft}$ of all measurable functions $\mathcal{M}(\mathbf{B})$ of Y into the closed unit interval $I = [0,1]$ into all measurable functions $\mathcal{M}(\mathbf{A})$ of X into I so that $T^{\triangleleft}$ has some natural properties. This way $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$, considered as bold algebras of fuzzy subsets of X resp. Y, become *fuzzy random events*, T becomes a *fuzzy random variable* and $T^{\triangleleft}$ becomes *fuzzy observable*. However, a degenerated point probability measure on $\mathbf{A}$ can be mapped to a nondegenerated probability measure on $\mathbf{B}$ and, consequently, fuzzy random variables and fuzzy observables do have genuine quantum and fuzzy properties. For example, a fuzzy observable, unlikely a classical observable, can map a crisp event (a set in $\mathbf{B}$) to a genuine fuzzy event (a function in $\mathcal{M}(\mathbf{A})$).

Example 3.1 Let q be a probability measure on $\mathbf{B}$ and let T be a map of $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ sending each $p \in \mathcal{P}(\mathbf{A})$ to q. Then for each $B \in \mathbf{B}$, the dual map T^d of $I^{\mathcal{P}(\mathbf{B})}$ into $I^{\mathcal{P}(\mathbf{A})}$ sends the evaluation $(\chi_B)^*$ of the characteristic function χ_B of B to the evaluation $v^* \in I^{\mathcal{P}(\mathbf{A})}$ of $v \in \mathcal{M}(\mathbf{A}))$ defined by $v(x) = q(B), x \in X$. Hence (cf. Theorem 2.4 in [10]) T is measurable. If $0 < q(B) < 1$, then the observable $T^{\triangleleft}$ sends a crisp event χ_B to a fuzzy event v. Observe that T generalizes a classical degenerated random variable f_r sending each elementary event $\omega \in \Omega$ to a given $r \in R$.

Now, let $(X, \mathbf{A})$, $(Y, \mathbf{B})$ be classical measurable spaces, let $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$ be the bold algebras of all measurable functions of X and of Y into I, respectively. Given a natural number n, let $\mathcal{X} \subseteq S_n^X$ be the simple S_nD-domain having $\mathcal{M}(\mathbf{A})$ as a base and let $\mathcal{Y} \subseteq S_n^Y$ be the simple S_nD-domain having $\mathcal{M}(\mathbf{B})$ as a base. Let X^* and Y^* be the sets of all sequentially continuous S_nD-homomorphisms of $\mathcal{X}$ and of $\mathcal{Y}$ into S_n, respectively. Let $\mathcal{X}^*$ and $\mathcal{Y}^*$ be the corresponding sobrifications and let T be an S_nD-measurable map of $(X^*, \mathcal{X}^*)$ into $(Y^*, \mathcal{Y}^*)$. Since $\mathcal{X}^*$ and $\mathcal{X}$, resp. $\mathcal{Y}^*$ and $\mathcal{Y}$, are S_nD-isomorphic, we can consider the dual map $T^{\triangleleft}$ as a sequentially continuous S_nD-homomorphism of $\mathcal{Y}$ into $\mathcal{X}$. Under the present notation, in the next example we indicate that the notion of a degenerated random variable can be generalized one step further.

Example 3.2 Let s be a state on $\mathcal{M}(\mathbf{B})$, let s_n be the corresponding sequentially continuous S_nD-homomorphism of $S_n(\mathcal{M}(\mathbf{B}))$ into S_n based on s. Define $T : X^* \to Y^*$ by $T(x) = s_n$ for all $x \in X^*$. Then T is measurable. Indeed, for $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in S_n(\mathcal{M}(\mathbf{B}))$ and $x \in X$ we have $(T(x))(\mathbf{f}) = s_n(\mathbf{f}) = (s(f_1), s(f_2), \ldots, s(f_n))$. But this means that $T^{\triangleleft}$ (as a map of $S_n(\mathcal{M}(\mathbf{B}))$) sends $\mathbf{f}$ to a "coordinatewise constant" element of S_n^X. Since such elements belong to $S_n(\mathcal{M}(\mathbf{A}))$, T is measurable. Clearly, if $0 < s(B) < 1$ for some $B \in \mathbf{B}$, then $T^{\triangleleft}((\chi_B, 0, \ldots, 0)) = (c, 0, \ldots, 0)$, where c is a constant function such that $c(x) = s(B)$ for all $x \in X$, hence $T^{\triangleleft}$ maps a "crisp" element $(\chi_B, 0, \ldots, 0)$ of $S_n(\mathcal{M}(\mathbf{B}))$ to a "noncrisp" element $(c, 0, \ldots, 0)$ of $S_n(\mathcal{M}(\mathbf{A}))$.

Conclusion It is natural to define basic generalized probability notions within the category S_nD as follows:

- random events ... a simple S_nD-domain $\mathcal{X} \subseteq S_n^X$;
- state ... a sequentially continuous S_nD-homomorphism into S_n;
- observable ... a sequentially continuous S_nD-homomorphism of a simple S_nD-domain $\mathcal{Y} \subseteq S_n^Y$ into a simple S_nD-domain $\mathcal{X} \subseteq S_n^X$;
- random variable ... a measurable map of the sobrification $(X^*, \mathcal{X}^*)$ of a S_nD-measurable space $(X, \mathcal{X})$ into the sobrification $(Y^*, \mathcal{Y}^*)$ of a S_nD-measurable space $(Y, \mathcal{Y})$.

In the special case described in the previous Example, the resulting notions have a generalized fuzzy and quantum nature. Incentives for further research can be found e.g. in [11, 13].

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